

Algebraic entropy for semi-discrete equations

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March 7, 2013

Abstract

We extend the definition of algebraic entropy to semi-discrete (difference-differential) equations. Calculating the entropy for a number of integrable and non integrable systems, we show that its vanishing is a characteristic feature of integrability for this type of equations.

1 Introduction

Algebraic entropy [1] was introduced as a measure of the complexity of discrete systems, originally for maps over finite dimensional spaces and finite order discrete (difference) equations. It relates to general notions of complexity of maps [2–13]. Its use as an integrability detector for such systems is by now firmly established [14–17]. Its definition has been further extended to infinite-dimensional systems, that is to say partial difference equations [18, 19], and it provides a very good characterisation of integrability for these systems as well [20]. In all cases integrability causes a drop in complexity, making the entropy vanish, while the entropy is nonzero for generic non integrable systems.

We give in Section 2 a generalisation of the concept to systems which mix difference and differential equations. These semi-discrete equations constantly appear, for example from partial discretisation of continuous systems or in the description of symmetries of integrable equations, and the "low growth requirement" has already been used for such systems, in conjunction with singularity confinement in the earlier Ref. [21]. In Section 3 we recall the link between the singularity structure and the value of the entropy. In Section 4 we describe how to calculate the entropy.

In Section 5 we calculate the entropy for two discretisations of the nonlinear Schrödinger equation: the integrable one given by the Ablowitz-Ladik system [22] and a more naive non integrable one. In Section 6 we calculate the entropy of Yamilov's form [23] of the discrete Krichever-Novikov equation (see also Ref. [24]). In Section 7 we apply the definition to nonautonomous systems, the master symmetries of the Volterra, modified Volterra, and discrete Calogero-Degasperis equations given in Ref. [25].

The outcome of all our calculations is that a vanishing entropy is, here again, a characteristic feature of integrability.

We conclude by indicating some directions for further studies.

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2 Definition

We deal with difference-differential equations of the form

$$u_{n+1}(t) = \frac{A + B u_{n-k}(t)}{C + D u_{n-k}(t)}, \quad (1)$$

where t is a ν -tuple of “times” $[t_1, t_2, \dots, t_\nu]$ and A, B, C, D are rational functions of $u_n(t), \dots, u_{n-k+1}(t)$ and a finite number of their derivatives $\partial_{t_q} u_n, \partial_{t_q t_r}^2 u_n, \dots, \partial_{t_q} u_{n-k+1}, \partial_{t_q t_r}^2 u_{n-k+1}, \dots$, with respect to t_1, t_2, \dots, t_ν . The functions A, B, C, D may explicitly depend on the times t_1, t_2, \dots, t_ν and on the discrete index n (non autonomous case).

Such equations define invertible recurrences of order $k + 1$, that is to say a very large number of interesting equations.

The first step is to define from Eq. (1) a homogeneous map by projectivisation. The simplest presentation of this process is to write

$$\begin{bmatrix} u_n & = & X_1/X_{k+1} \\ u_{n-1} & = & X_2/X_{k+1} \\ & \dots & \\ u_{n-k+1} & = & X_k/X_{k+1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} u_{n+1} & = & Y_1/Y_{k+1} \\ u_n & = & Y_2/Y_{k+1} \\ & \dots & \\ u_{n-k+2} & = & Y_k/Y_{k+1} \end{bmatrix}.$$

Equation (1) defines a map from $(u_n, u_{n-1}, \dots, u_{n-k+1})$ to $(u_{n+1}, u_n, \dots, u_{n-k+2})$ which translates into a map φ from the $(k+1)$ -tuple $x = [X_1, X_2, \dots, X_{k+1}]$ to its image $y = [Y_1, Y_2, \dots, Y_{k+1}]$. All components of y are polynomials in X_1, X_2, \dots, X_{k+1} and a finite number of their derivatives with respect to the times t_1, t_2, \dots, t_ν . If we attribute to all X ’s and any of their derivatives a weight 1, and weight 0 to all coefficients, even if they depend on the times or the discrete indices, these polynomials are homogeneous with some weight w_1 .

We then iterate the map φ . Since we have homogenised the coordinates, any factor common to the $k+1$ components should be factored out. Doing so we generate a sequence of $(k+1)$ -tuples, all with a well defined weight (using the simple fact that derivatives with respect to the times satisfy Leibnitz’ rule). This defines a sequence of weights w_1, w_2, \dots .

The sequence $\{w_m\}$ satisfies the property

$$w_{l+m} \leq w_l \cdot w_m. \quad (2)$$

Definition: The entropy is defined as

$$\epsilon = \lim_{m \rightarrow \infty} \frac{1}{m} \log(w_m). \quad (3)$$

Claim: The previous property Eq. (2) of the sequence of weights ensures the existence of this limit.

3 Singularity structure and entropy

There is a deep link between the singularity structure [26] of the evolution and the entropy. The defining equation (1) is such that the map φ has a polynomial inverse ψ . This allows us to define two polynomials κ_φ and κ_ψ as in Ref. [1]: since φ and ψ are inverse of each other, their product appears as a mere multiplication of the components by some polynomial

$$(\psi \cdot \varphi)(x) = \kappa_\varphi(x).id(x) \quad \text{and} \quad (\varphi \cdot \psi)(x) = \kappa_\psi(x).id(x),$$

where id is the identity map.

During the iteration of φ , the appearance of some factor F common to all components means that the hypersurface given by $F = 0$ is sent by the corresponding iterate of φ on a singular set of φ (singularity means that the polynomial image is $[0, 0, \dots, 0]$). The only possible factors are thus the factors composing κ_φ and their proper images by ψ . We will give some examples of this in the sequel.

If there is no factorisation, the sequence of weights is purely exponential and

$$w_n = w_1^n, \quad \epsilon = \log(w_1).$$

Claim: Any drop in degree during the iteration comes from the possible presence of common factors, and is related to the singularity structure for the geometrical reason explained above.

We will illustrate this fundamental property in some of the examples studied below.

4 Actual calculations

One may of course calculate the iterates of φ . The main obstacle is the size of the explicit expressions, which rapidly exceeds the capacity of hand calculations as well as computer-aided ones.

The first issue is that we work with an algebra generated by an infinite-dimensional set: the X 's and their derivatives. One should however notice that, at every finite order, one uses only a finite subset of the generators, and manipulates only polynomials. This makes the calculation of a few of the first terms of the sequence $\{w_m\}$ possible.

To go further we will use the method described in Ref. [27]. We calculate the iterates of φ starting with a rational initial condition, of the form $[X_1(t_1, t_2, \dots, t_\nu), \dots, X_{k+1}(t_1, t_2, \dots, t_\nu)]$, with all the X 's polynomials in the times appearing in Eq. (1).

One then produces the sequence of degrees $\{d_n\}$ of the iterates. The point is that, *for generic initial conditions, this sequence retains the feature of the sequence of weights $\{w_m\}$ we are interested in: its asymptotic behaviour.* In some cases, and with a judicious choice of initial conditions, the two sequences $\{d_n\}$ and $\{w_m\}$ may coincide exactly.

The next step is to extract the value of the entropy, which is an asymptotic quantity, from the first few terms of the sequence of weights (resp. degrees). The standard heuristic method is to fit a generating function of the sequences with rational fractions.

$$g_w(s) = \sum_{n=0}^{\infty} w_n s^n = \frac{P_w(s)}{Q_w(s)}, \quad \text{or} \quad g_d = \sum_{n=0}^{\infty} d_n s^n = \frac{P_d(s)}{Q_d(s)}. \quad (4)$$

The method has already been shown to work remarkably well for maps and lattice equations [17, 20], and leads to extremely simple rational fractions with integer coefficients. This is related to the algebraic properties of the entropy, in particular the still conjectured fact that the entropy always is the logarithm of an algebraic integer [1, 28], but we will not dwell on that here.

Once the first terms of the sequence are fitted with the Taylor coefficients of a rational fraction, the method becomes predictive, and is a good check of the validity of the generating function. The location of the smallest pole of the denominator ($Q_d(s)$ or $Q_w(s)$) of the generating function gives the exact value of the entropy.

Remark 1: When the entropy vanishes, the growth of the sequences is polynomial. A quick test is to calculate the successive discrete derivatives of the sequences. Linear growth is equivalent to a bounded first derivative, quadratic growth is equivalent to a bounded second derivative, and so on.

Remark 2: The exact value of the entropy may sometimes be extracted from the singularity analysis. In one of the cases we will be able to do that (see Section 5.2).

Remark 3: It is quite possible to perform the calculation of the iterates without going to the homogeneous description. The expressions are then rational fractions rather than polynomials, and the degree to retain is the maximum degree of their numerators and denominators. The sequence of degrees one produces is usually different from the one given by the homogeneous form. It nevertheless yields the same value of the entropy, since the discrepancy between the two sequences is at most a multiplicative factor smaller than or equal to the order $k + 1$.

5 Two semi-discrete non linear Schrödinger equations

One source of semi-discrete equations is the incomplete discretisation of partial differential equations. The nonlinear Schrödinger equation is a typical example of such systems:

$$iu_t + u_{xx} + 2|u|^2 u = 0,$$

with u a complex function of a space variable x and a time variable t . If only the spatial coordinate x is discretised, we get a semi-discrete equation with one continuous time variable ($\nu = 1$). We will examine two cases. The first is the integrable one given by Ablowitz and Ladik [22]:

$$i \partial_t u_j + (u_{j-1} - 2u_j + u_{j+1})/h^2 + |u_j|^2(u_{j-1} + u_{j+1}) = 0. \quad (5)$$

The second one is a slightly more naive one

$$i \partial_t u_j + (u_{j-1} - 2u_j + u_{j+1})/h^2 + 2|u_j|^2 u_j = 0. \quad (6)$$

Here h is the discretisation step. Both cases are perfectly sensible as discretisations of Eq. (5). However they have been shown to behave very differently [29], the reason being that only the form Eq. (5) retains one of the main features of the original equation: its integrability. We will evaluate their respective entropies.

5.1 Ablowitz-Ladik system

We rewrite equation (5) as a system for the real and imaginary parts q_j and r_j of u_j , with some rescaling:

$$\begin{aligned} \partial_t q_n &= q_{n+1} - 2q_n + q_{n-1} + q_n r_n (q_{n+1} + q_{n-1}) \\ -\partial_t r_n &= r_{n+1} - 2r_n + r_{n-1} + q_n r_n (r_{n+1} + r_{n-1}). \end{aligned} \quad (7)$$

The homogenisation proceeds by setting

$$q_n = \frac{X}{V}, \quad r_n = \frac{Y}{V}, \quad q_{n-1} = \frac{Z}{V}, \quad r_{n-1} = \frac{U}{V}.$$

The map φ in the homogeneous coordinates $[X, Y, Z, U, V]$ reads

$$\varphi : \begin{pmatrix} X \\ Y \\ Z \\ U \\ V \end{pmatrix} \longrightarrow \begin{pmatrix} V^2(X' + 2X - Z) - X(VV' + YZ) \\ V^2(-Y' + 2Y - U) + Y(VV' - XU) \\ X(V^2 + XY) \\ Y(V^2 + XY) \\ V(V^2 + XY) \end{pmatrix}$$

which is a degree 3 birational map over the algebra generated by

$$X, X', X'', \dots, Y, Y', Y'', \dots, Z, Z', Z'', \dots, V, V', V'' \dots$$

where $X' = \partial_t X$, etc.

The inverse is

$$\psi : \begin{pmatrix} X \\ Y \\ Z \\ U \\ V \end{pmatrix} \longrightarrow \begin{pmatrix} Z(V^2 + UZ) \\ U(V^2 + UZ) \\ V^2(Z' + 2Z - X) - Z(VV' + UX) \\ V^2(-U' + 2U - Y) + U(VV' - YZ) \\ V(V^2 + UZ) \end{pmatrix}.$$

The direct calculation of the iterates of φ produces

$$\{w_n\} = 1, \quad 3, \quad 9, \quad 19$$

which is not long enough to guess the asymptotic behaviour.

The specialisation of the initial condition with linear functions of t , which provides the simplest nontrivial rational initial values for q_n, r_n , produces the sequence

$$\{d_n\} = 1, 3, 9, 19, 33, 51, 73, 99, 129, 163, 201, 243, 289, \dots$$

The sequence is (redundantly) fitted by the generating function

$$g(s) = \frac{1 + 3s^2}{(1 - s)^3},$$

showing quadratic growth, vanishing entropy, i.e. integrability.

Consider degree drop and singularity pattern. The first drop in degree comes with φ^3 , since $3 \times 9 - 19 = 8$. We know that this happens when φ hits some singular point. The pattern is the following: the surface $\Sigma_+ : V^2 + XY = 0$ is sent (blown down) to $[-X^2, V^2, 0, 0, 0]$ by φ . At the next step it goes to $[0, 0, X^2, -V^2, 0]$, a singular point which φ blows up to the surface $\Sigma_- : V^2 + UZ = 0$. The third iterate sends “smoothly” Σ_+ into Σ_- , and the drop of the degree is due to the presence of the factor Σ_+^4 .

5.2 Naive discretisation of the nonlinear Schrödinger equation

Consider now equation (6) in terms of the same variables q_n, r_n :

$$\begin{aligned} \partial_t q_n &= q_{n+1} - 2q_n + q_{n-1} + 2q_n^2 r_n \\ -\partial_t r_n &= r_{n+1} - 2r_n + r_{n-1} + 2q_n r_n^2. \end{aligned} \tag{8}$$

With the same notations as in the previous case, the map φ reads

$$\varphi : \begin{pmatrix} X \\ Y \\ Z \\ U \\ V \end{pmatrix} \longrightarrow \begin{pmatrix} ZV^2 \\ UV^2 \\ (2Z - X)V^2 - 2Z^2U + Z'V^2 - ZVV' \\ (2U - Y)V^2 - 2ZU^2 - U'V^2 + UVV' \\ V^3 \end{pmatrix}.$$

The sequence of degrees we get is purely exponential

$$\{d_n\} = 1, 3, 9, 27, 81, 243, \dots$$

It is straightforward to prove that the sequence of weights is exactly the same; that is, there is no degree drop, $w_n = 3^n$.

The inverse of ψ is related to φ by a linear similarity. If λ is the permutation

$$\lambda : [X, Y, Z, U, V] \longrightarrow [Z, U, X, Y, V],$$

then

$$\psi = \lambda \cdot \varphi \cdot \lambda.$$

The polynomials κ_φ and κ_ψ are just

$$\kappa_\varphi = \kappa_\psi = V^8.$$

The ‘‘hypersurface’’ $V = 0$ is sent by φ to one of its fixed points $[0, 0, Z, U, 0]$. Consequently the later iterates cannot hit any singular point, and there will be no factorisation. This proves that the entropy is exactly $\epsilon = \log(3)$ in contrast with the vanishing entropy of the integrable discretisation of Ref. [22].

6 Generalised symmetries of integrable equations

A source of integrable semi-discrete equations is the symmetry approach to integrability of discrete systems [30–36]. A fundamental equation is Yamilov’s form of the discrete Krichever-Novikov equation [23]. The equation is

$$\partial_t u_n = \frac{R(u_{n+1}, u_n, u_{n-1})}{u_{n+1} - u_{n-1}}, \quad (9)$$

with six free parameters β_i , $i = 1 \dots 6$:

$$\begin{aligned} R(u_{n+1}, u_n, u_{n-1}) = & (\beta_1 u_n^2 + 2\beta_2 u_n + \beta_3) u_{n+1} u_{n-1} + (\beta_2 u_n^2 + \beta_4 u_n + \beta_5) (u_{n+1} + u_{n-1}) \\ & + \beta_3 u_n^2 + 2\beta_5 u_n + \beta_6. \end{aligned} \quad (10)$$

Using an obvious notation, the homogeneous version of the corresponding map reads

$$\varphi : \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \longrightarrow \begin{pmatrix} -Z [\beta_2 X^2 Y + \beta_3 X^2 Z + \beta_4 X Y Z + \beta_5 Z^2 (Y + 2X) + \beta_6 Z^3 + Y(X'Z - XZ')] \\ X [\beta_1 X^2 Y + \beta_2 X Z (2Y + X) + \beta_3 Y Z^2 + \beta_4 X Z^2 + \beta_5 Z^3 - Z(X'Z - XZ')] \\ Z [\beta_1 X^2 Y + \beta_2 Z X (2Y + X) + \beta_3 Y Z^2 + \beta_4 X Z^2 + \beta_5 Z^3 - Z(X'Z - XZ')] \end{pmatrix},$$

The sequence of degrees obtained with degree 1 initial conditions is:

$$\{d_n\} = 1, 4, 10, 20, 34, 52, 74, 100, 130, 164, 202, 244, 290, 340, 394, 452, \dots$$

fitted by the generating function

$$g(s) = \frac{(1+s)(1+s^2)}{(1-s)^3},$$

meaning quadratic growth and vanishing entropy.

Consider the singularity structure. We have

$$\kappa_\varphi = A^2 B^3 C,$$

with

$$\begin{aligned} A &= Z, \quad B = \beta_1 X^2 Y + \beta_2 X Z (2Y + X) + \beta_3 Y Z^2 + \beta_4 X Z^2 + \beta_5 Z^3 + Z(-X'Z + XZ') \\ C &= 2XZ [(\beta_1 \beta_5 + \beta_2 \beta_3 - \beta_2 \beta_4) X^2 + (\beta_2 \beta_5 + \beta_3^2 - \beta_4^2 + \beta_1 \beta_6) XZ + (\beta_2 \beta_6 - \beta_4 \beta_5 + \beta_3 \beta_5) Z^2] \\ &\quad + (\beta_1 \beta_3 - \beta_2^2) X^4 + (\beta_3 \beta_6 - \beta_5^2) Z^4 + (XZ' - X'Z)^2. \end{aligned}$$

For example, at step φ^2 the product $A^3 B$ factors out causing a degree drop of $6 = 4 \times 4 - 10 = 3 + 3$.

At the next step the factor is $B^3 C \varphi_*(B)$ causing a degree drop of $20 = 4 \times 10 - 20 = 3 \times 3 + 4 + 7$ (here $\varphi_*(B) = 0$ is the equation of the proper image by ψ of the surface of equation $B = 0$, i.e. of the pull-back by φ). So the geometrical interpretation of the degree drops is perfectly in order.

Remark: Changing in an arbitrary way the expression of R completely changes the result: for example introducing a new independent parameter β_7 as the coefficient of $u_n^2(u_{n-1} + u_{n+1})$ in Eq. (10) leads to a sequence of degrees $1, 4, 10, 24, 58, 140, 338, 816, 1970, \dots$, fitted by $g(s) = (1+s)^2/(1-2s-s^2)$, meaning an entropy of $\log(1+\sqrt{2})$.

7 Non autonomous systems

Three interesting semi-discrete equations are provided by master symmetries of the Volterra, modified Volterra, and discretised Calogero-Degasperis equations, respectively Eqs. (6), (20) and (26) of Ref. [25].

- The first equation is

$$\partial_t u_n = u_n [(\epsilon + n + 2) u_{n+1} + u_n - (\epsilon + n - 1) u_{n-1}]. \quad (11)$$

It is nonautonomous in the discrete variable n , but the coefficient ϵ is constant.

The corresponding map reads

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \longrightarrow \begin{pmatrix} X'Z - XZ' - X^2 + (\epsilon + n - 1)XY \\ (\epsilon + n + 2)X^2 \\ (\epsilon + n + 2)XZ \end{pmatrix}$$

where $X' = \partial_t X$, etc...

One finds the sequence of degrees

$$\{d_n\} = 1, 2, 4, 8, 13, 20, 28, 38, 49, 62, 76, 92, 108, 127, 146, 166, 188, 211, 236, 262, 290, 319, 350, \dots$$

fitted by the generating function

$$g(s) = \frac{(1 + 2s^3)(1 - s^{12} + s^{13})}{(1 + s)(1 - s)^3}.$$

We have quadratic growth of the degree, vanishing entropy, and integrability.

- The second equation is given by

$$\partial_t v_n = (\lambda - v_n) v_n [(\epsilon + n + 1) v_{n+1} - (\epsilon + n - 1) v_{n-1}] + \lambda v_n (v_n - \lambda/2). \quad (12)$$

Here the parameter λ *depends on the time* and satisfies the condition

$$\partial_t \lambda = \lambda^3/2. \quad (13)$$

Equation (12) is thus non autonomous both in the discrete and the continuous directions.

For this case, the map φ is

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \longrightarrow \begin{pmatrix} 2Z(X'Z - XZ') + 2XY(\lambda Z - X)(\epsilon + n - 1) + \lambda XZ(\lambda Z - 2X) \\ 2X^2(\lambda Z - X)(\epsilon + n + 1) \\ 2XZ(\lambda Z - X)(\epsilon + n + 1) \end{pmatrix}.$$

We get the sequence

$$\{d_n\} = 1, 3, 9, 19, 33, 51, 73, 99, 129, \dots$$

which has quadratic growth: the second derivative is constant (value 4).

Notice that if we do not impose the correct time dependence of λ given by Eq. (13), supposing that λ is constant, the entropy does not vanish anymore. We get the sequence 1, 3, 9, 22, 51, 116, 262, 590, 1327, ... fitted by the generating function

$$g(s) = \frac{1 + s^2 - s^4}{(1 - s)(s^3 - s^2 - 2s + 1)}$$

and giving an entropy of approximately $\log(2.246979604)$ [logarithm of the inverse of the smallest root of $(s^3 - s^2 - 2s + 1)$].

- The third equation reads

$$4 \partial_t v_n = (1 - v_n^2) \left[(b^2 - a^2 v_n^2) \left(\frac{\epsilon + n}{v_{n+1} + v_n} - \frac{\epsilon + n - 1}{v_n + v_{n-1}} \right) + a^2 v_n \right], \quad (14)$$

with $a = \lambda + \mu$ and $b = \lambda - \mu$, the two coefficients λ and μ verifying the same equation (13).

We will not write explicitly the formula for the map φ here.

We get the sequence

$$\{d_n\} = 1, 4, 13, 28, 49, \dots$$

which has constant second derivative (value 6).

If one took a and b constant, forgetting their time dependence, the sequence of degrees would be 1, 4, 15, 42, 107, 264, 643, ... showing exponential growth, and non vanishing entropy $\log(1 + \sqrt{2})$.

8 Conclusion

All our calculations, without exception, show that the vanishing of the entropy is the hallmark of integrability for the extended class of semi-discrete systems, as it was for the purely discrete ones.

Our study can be complemented in the future in a number of directions:

- In order to completely validate the calculations made with degrees rather than weights, it is sufficient to examine the factors dropping from the successive images under the iteration, and check that all factors are in agreement with the geometrical description given in Section 3. This will guarantee the absence of spurious factorisation, due to the possible non-genericity of the initial conditions.
- We have not described here equations with more than one time ($\nu \geq 2$), as for example the 2-dimensional infinite Toda field theory

$$\partial_{xt}^2 u_n = \exp(u_{n+1} - 2u_n + u_{n-1})$$

which can be turned into an equation of the form (1) by the change of variable $\exp(u_n) = v_n$. Our definition applies directly to this type of higher dimensional systems, and the result is the same.

- Another direction to explore is lattice equations, that is to say systems with more than one discrete index, to which the definition can be adapted.
- Finally, the homogeneous setting we have introduced is more general than the recurrence (1) we started from. The definition we gave can thus be applied to a much wider class of semi-discrete equations.

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